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Note

On Summability and Positive Linear Operators

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Quantitative estimates for approximation by positive linear operators are obtained with the use of a summability method which includes both convergence and almost convergence.

Korovkin's famous theorem [5] regarding convergence of sequences of positive linear operators in the space of continuous functions was put into a quantitative form by Shisha and Mond [8]. In [4] it was shown that Korovkin's results are valid if convergence is replaced by almost convergence, and the modified results were recently put into quantitative form by Mohapatra [7]. It is the purpose of this note to bring some unification through the use of a summability method introduced by H. T. Bell [1].

Let $B = \{A^{(n)}\} = \{(a_{kj}^{(n)})\}$ be a sequence of infinite matrices such that $a_{kj}^{(n)} \geqslant 0$ for $k, j, n = 1, 2,\ldots$. A sequence of real numbers, $\{x_j\}$, is said to be B summable to L if

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} x_j = L$$

uniformly in $n = 1, 2,\ldots$.

If, for some matrix A, $A^{(n)} = A$ for $n = 1, 2,\ldots$, then B summability is just matrix symmabitlity by A. If, for $n = 1, 2,\ldots$, $a_{kj}^{(n)} = 1/k$ for $n \leqslant j < k + n$, and $a_{kj}^{(n)}$ is 0 otherwise, then B summability reduces to almost convergence [6]. We also note that the method of order summability of Jurkat and Peyerimhoff [2, 3] is a special case of B summability [1].

Let $\{L_j\}$ be a sequence of positive linear operators from $C[a, b]$ to $C[a, b]$ and let $\{A^{(n)}\} = B$ be a sequence of infinite matrices with non-negative real entries. For $f \in C[a, b]$, $A^{(n)}(f, x)$ denotes the double sequence

$$A_k^{(n)}(f, x) = \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f(t), x), \qquad k, n = 1, 2,\ldots .$$

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We define $\| A_k(f)\|$ to be

$$\sup_n \sup_{x\in[a,b]} | A_k^{(n)}(f, x)|$$

and we assume that

$$\| A_k(C_0)\| < \infty \tag{1}$$

where $C_0(x) = 1$ for all $x \in [a, b]$. It then follows that, for $f \in C[a, b]$, $\{L_j(f)\}$ is B summable to f, uniformly on $[a, b]$, if and only if

$$\| A_k(f) - f\| \equiv \sup_n \sup_{x\in[a,b]} | A_k^{(n)}(f) - f(x)|$$

tends to 0 as k tends to ∞.

The proofs of the following theorems, which are similar to the proofs of the corresponding results of [7] and [8], are omitted.

THEOREM 1. *Let $\{L_j\}$ be a sequence of positive linear operators from $C[a, b]$ to $C[a, b]$. Let $B = \{A^{(n)}\}$ be a sequence of infinite matrices with non-negative real entries. Assume* (1) *is satisfied. Then, for $f \in C[a, b]$ and $k = 1, 2,\ldots,$*

$$\|f - A_k(f)\| \leqslant \|f\| \cdot \| A_k(e_0) - 1 \| + w(\mu_k) \| A_k(e_0) + 1 \|$$

where

$$\mu_k{}^2 = \| A_k((t - x)^2)\|,$$

$$\|f\| = \sup_{x\in[a,b]} |f(x)|,$$

and w denotes the modulus of continuity of f.

Let K be the additive Abelian group of real numbers modulo 2π on which the metric d is defined by

$$d(x, y) = \min\{| x - y |, 2\pi - | x - y |\},$$

for $x, y \in K$, $0 \leqslant x, y \leqslant 2\pi$. Let $C(K)$ denote the set of all continuous, real valued functions on K. For $f \in C(K)$, the modulus of continuity, w, is defined by

$$w(f, \delta) = \sup_{\substack{x,y\in K\\ d(x,y)\leqslant\delta}} |f(x) - f(y)|. \tag{2}$$

THEOREM 2. *Let $\{L_j\}$ be a sequence of positive linear operators from $C(K)$ to $C(K)$. Assume* (1) *holds with $[a, b]$ replaced by K, where $B = \{A^{(n)}\}$*

is a sequence of infinite matrices with nonnegative real entries. Then, for $f \in C(K)$ *and* $k = 1, 2, \ldots$,

$$\| A_k(f) - f \| \leqslant \| f \| \cdot \| A_k(e_0) - 1 \| + w(\mu_k) \| A_k(e_0) + 1 \|$$

where w *is defined by* (2),

$$\| f \| = \sup_{x \in K} | f(x) |,$$

and

$$\mu_k^2 = \| A_k(\sin^2((t - x)/2)) \|.$$

We also note that results analogous to Theorems 3 and 4 of [7] can be obtained for B summability.

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